DORFMAN AND R_1-TYPE PROCEDURES FOR A GENERALIZED GROUP-TESTING PROBLEM

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ABSTRACT

The enclosed paper is a basic study of the Dorfman procedure for a one-test problem and shows how to apply new and old results to a generalized group-testing problem in which each unit has to be positive on two independent tests to be classified as defective. Procedures of type R_1 (cf. [6]) are also developed. The procedures developed are compared, but large scale numerical results have not yet been obtained.

These procedures can be applied to different areas of medical research. Suppose, for example, that a test for cancer lacks specificity and a female patient shows a positive reaction if she has cancer or if she is pregnant. Then it becomes necessary to carry out a second test and we assume that i) two positive reactions for a single individual is a definite indication of cancer, although the second test may also not be specific, and ii) both tests can be carried out by pooling speciments of any number of patients. As in the original group-testing problem any positive reaction on a sample pooled from x patients indicates only that at least one positive is present and a negative reaction indicates that all x are negative.

In the original group-testing problem with one test savings of 80 percent in sampling costs relative to the one-at-a-time procedure are possible when p (the probability that a unit is defective) is .01 and for the R_1-procedure the saving relative to the one-at-a-time procedure is over 90 percent. Similar and even better results are expected for the procedures of this paper. For example, with N = 100 patients and $p = .01$ the one-at-a-time procedure requires 200 tests whereas the R_1 procedure requires an average of 8.320 tests for tests of the first type

and (after getting an average of $Np_j = 1$ positive) about 1 test of the second type; this makes less than 10 tests on the average. Thus we have a saving of about 190/200 or 95 percent; a more exact computation in Table 1 of Section 5 gives the saving in this case as 95.52 percent. This table also includes some values for the Dorfman procedure R^D_{12} for $N = 100$ and lower bounds based on (5.1).

1. Introduction.

In group-testing the basic problem is to classify each of N units into two disjoint categories: good or defective. The characteristic property of the group-test is that we can test any number of units, say x, simultaneously ($1 \leq x \leq N$) as a group with one of 2 possible outcomes: either all x are good or at least one defective unit is present (we don't know which ones or how many). The basic problem which was treated in [2], [6] and [4] is to find the optimum sample size x for the first stage and for each of the subsequent stages (the latter may depend on results already observed), in order to minimize the expected total number of group-test required to classify each of the N units. N is assumed to be large and finite and for some procedures can be taken as infinite; in the latter case and perhaps also for N finite we use the expected number of tests per unit classified as a criterion of efficiency. It is assumed that the N units can be treated (at the outset) like independent binomial chance variables with a common known probability p of being defective. The case of p unknown and the case of p generated by a prior distribution have also been treated (cf. [6] and [7]).

In this paper we assume that two different tests (T_1 and T_2) can be made on groups of any size (or sizes) and that the result for each test is again dichotomous: positive (+) or negative (-). We define a unit to be defective if (and only if) it is positive on both T_1 and T_2. For an arbitrarily chosen unit let X_i be one or zero according as it is + or - on test T_i ($i = 1, 2$). We assume that the random variables X_1 and X_2 are independent for the same unit and also for was considered in [1] by assuming that each group-test has three possible outcomes: all good, all bad, or at least one of each; however, the asymmetric model treated in this paper appears to have more applications.

In another application (that we may call 'Lack of Specificity') we deal with three distinct diseases, D_1, D_2 and D_3. One test, say T_1, gives a positive result when either D_1 or D_2 or both are present and the other test T_2 gives a positive result when either D_1 or D_3 or both are present. We wish to cull out all patients with disease D_1. Our first job is to separate out all the units of $(+, +)$ type and if the pair (D_2, D_3) can coexist in a patient without D_1 being present then our subsequent problem is to remove these false positives from the $(+, +)$ set. In any case the first part (separating out the $(+, +)$ patients) does fall within our present formulation.

Some of the notation in [6] will be used here also. In particular, let $H_i(n)$ denote the expected number of tests required to complete procedure R_i if n units are currently still unclassified and the joint à posteriori distribution of their dichotomous states is a product of independent binomial distributions; at the outset this assumption holds with $n = N$.

2. Dorfman Procedure for a Single Type Test.

Before defining the two generalized Dorfman procedures for our problem we define and give some background and new results for the simple Dorfman procedure as it applies to a single type test and for all values of N (including $N = \infty$).

The Dorfman procedure for a single type test [2] partitions the N given units into groups of equal (or approximately equal) size r

and treats each group separately as follows: "Test a group of size r. If good, go to the next group. If bad, test each unit of that group separately." At this point it is assumed that N is large or that it is an exact multiple of the group size r. The optimal group size r (which we later denote both by r and by r_0) is determined to minimize the relative cost (i.e., the expected number of tests required to classify N units divided by N) and for large N and small p this value of r depends on p only. Hence, if N is finite and not a multiple of r the above phrase "approximately equal" does not clearly or uniquely define the partitioning aspect of the procedure. To define this more exactly we first consider an arbitrary common group-size r and let $g = [N/r]$ denote the integer number of groups of size r (we later determine the value of r that will optimize our criterion) and let $\theta = N - r[N/r]$ denote the remainder $(0 \leq \theta < r)$. If $\theta > 0$ then we treat the odd-sized group of size θ in one of the two following alternative ways, whichever yields the better result.

1) Redistribute the θ units among the g groups so that no two group sizes differ by more than one; this reduces the total number of groups to g.

2) Build up the odd-sized set by taking units from the g groups in such a way that no two sizes differ by more than one; in this case we have a total of $g + 1$ groups.

In general we use 1) for smaller θ-values and 2) for larger θ-values; the case $\theta = 0$ can be regarded as a special case of 1). For example, if $N = 13$ and $r = 5$ we would consider the partitions (6, 7) and (4, 4, 5) but not (3, 5, 5) and use that one of these two which gives a smaller expected number of tests; this would depend on p. The criteria of minimizing the relative cost and minimizing the expected number of tests will be seen below to be identical. Since we are in essence now using r only to define g and θ we should regard r as the arbitrary original or nominal group size, since we could end up with no group-size equal to r under either of the two above alternatives. Some justification for considering only these two alternatives will be given later for small p-values but we are now <u>defining</u> the Dorfman procedure for all N-values by considering only these two alternatives and using whichever one is better.

Let R_D^* denote the Dorfman procedure as defined above with r taken to optimize the relative cost. Under R_D^* (cf. Dorfman [2]) we do not use inference to save one test on the last unit of a bad group, if the first $r - 1$ units of that group are all good. Let R_D denote the modified Dorfman procedure that does make use of this inference. We are mostly concerned with the latter in this paper, although asymptotic $(p \to 0)$ results are the same for both.

If N is a multiple of an arbitrary nominal r (to be determined) then we can use (157) of [6] to show that for procedure R_D using alternative 1) above

$$(2.1) \qquad H_D(N) = N(1 - q^r + \frac{1-pq^{r-1}}{r}) = Nf(p, r) \text{ (say) } ;$$

this already shows that for large N the two criteria of minimizing the relative cost and minimizing the expected number of tests are equivalent. The only change for R_D^* is to remove $-pq^{r-1}$ and we denote the resulting expression in parentheses by $f^*(p, r)$.

For sufficiently large N the odd-sized group of size θ can be disregarded if we transfer our criterion from $H_D(N)$ to $H_D(N)/N$ and in that case the function $f(p, r)$ is of primary importance. We therefore defer any exact evaluation of $H_D(N)$ until after we examine $f(p, r)$. As in Dorfman [2] and Watson [8] we use asymptotic $(p \to 0)$ theory with large N (i.e., $N \to \infty$ as $p \to 0$) to approximate the optimal value of the nominal r. Differentiation of $f(p, r)$ with respect to r (and treating r as continuous) gives

$$\text{(2.2)} \qquad \frac{d}{dr}\left\{\frac{H_D(N)}{N}\right\} = f_r(p, r) = -q^r \ln q - \frac{pq^{r-1}\ln q}{r} - \frac{(1-pq^{r-1})}{r^2} .$$

A simple asymptotic $(p \to 0)$ analysis of (2.2) yields for procedure R_D the equation

$$\text{(2.3)} \qquad r^2 p[1 - rp + \binom{r}{2}p^2] = 1 - p$$

and the only change for R_D^* is to drop the last term, $-p$. This in turn yields the approximation to r_0

$$\text{(2.4)} \qquad r_0 \sim \frac{1}{\sqrt{p}}\left(1 + \frac{\sqrt{p}}{2} + \frac{cp}{8}\right)$$

where $c = -3$ for procedure R_D and $c = +1$ for procedure R_D^*; the first term of (2.4) was obtained for R_D^* in (3.10) of [8]. It is also interesting to point out that the asymptotic $(p \to 0)$ values of $H_D(N)$ and $H_D^*(N)$ are both given by

$$\text{(2.5)} \qquad H_D(N) \sim 2N\sqrt{p}\left\{1 - \frac{\sqrt{p}}{4} + \frac{(c-2)p}{8}\right\}$$

where c is as given above. The difference of the expectations for the two procedures is therefore approximated for small p by

$$\text{(2.6)} \qquad H_D^*(N) - H_D(N) \sim Np^{3/2}$$

The result (2.5) can also be written in the form

$$(2.7) \qquad H_D(N) \sim 2g\left\{1 + \frac{\sqrt{p}}{4} + \frac{(2c-3)}{8}\,p\right\}$$

where c is as given above for R_D and R_D^* (cf. Feller [3] p. 240, Vol. I, 3rd edition). This shows that the value of r_0 is such that each group requires about two tests on the average when p is small. In writing the above approximations we have assumed that N is large enough so that the numerical result is at least one; otherwise we use one as our approximation.

To partially justify our restriction to only the two alternative partitions described above, we study the convexity of $f(p, r)$ near $r = r_0$. The second derivative of $f(p, r)$ or of $f^*(p, r)$ with respect to r yields for $p \to 0$

$$(2.8) \qquad \frac{d^2}{dr^2}\left\{\frac{H_D(N)}{N}\right\} = f_{r,r}(p, r) \sim \frac{2}{r^3} \sim 2p^{3/2} > 0$$

for $r = r_0 \pm a$ with fixed a. This convexity at and near $r_0 > 1$ implies that if N is finite and not a multiple of r_0 then the result for any two group sizes (say, r_0 and $r_0 + 2$) that differ by two or more can be improved by replacing them both by $r_0 + 1$. Similarly for r_0 and $r_0 - 2$ and also for $r_0 - 1$ and $r_0 + 1$. Thus for p small we can restrict our attention to the two alternatives mentioned above in each of which the group sizes are determined by the fact that no two group sizes can differ by more than one.

We now return to a small sample analysis for $H_D(N)$; for the i^{th} alternative alone we shall denote the expected number of tests by $A_D^{(i)}(N)$ $(i = 1, 2)$. Under 1) with $r = r_0$ and g, θ as defined above and with $r' = [\frac{N}{g}]$ and $\theta' = N - gr'$ we obtain exactly

$$(2.9) \qquad A_D^{(1)}(N) = (g-\theta')r' \ f(p, r') + \theta'(r'+1)f(p, r'+1)$$

since there are θ' groups of size $r'+1$ and $g-\theta'$ groups of size r'. Under 2) we define $r'' = [N/(g+1)]$ and $\theta'' = N - (g+1)r''$ and obtain exactly

$$(2.10) \qquad A_D^{(2)}(N) = (g+1-\theta'')r''f(p, r'') + \theta''(r''+1)f(p, r''+1).$$

Hence, by our definition, we have exactly for all N and p

$$(2.11) \qquad H_D(N) = \mathrm{Min}\{A_D^{(1)}(N), A_D^{(2)}(N)\},$$

and a similar formula holds for $H_D^*(N)$ if f is replaced by f^*.

A lower bound for $H_D(N)$ is obtained by getting lower bounds in (2.9) and (2.10) separately and using their minimum. If we replace both r' in (2.9) and r'' in (2.10) by r $(= r_0)$ then we obtain

$$(2.12) \qquad H_D(N) \geq f(p, r)\mathrm{Min}\{gr' + \theta', (g+1)r'' + \theta''\} = Nf(p, r)$$

since $gr' + \theta' = (g+1)r'' + \theta'' = N$. Hence the basic large-N approximation in (2.1) for small p is a lower bound for all N and all p.

An upper bound for $H_D(N)$ is obtained similarly, using the fact that $rf(p, r)$ is increasing in r. We obtain

$$(2.13) \qquad H_D(N) \leq \mathrm{Min}\{g(r'+1)f(p, r'+1), (g+1)(r''+1)f(p, r''+1)\}$$
$$\leq (g+1)(r+1)f(p, r+1) \sim (g+1)rf(p, r) \sim 2g + 2$$

since $r'' \leq r$ and $rf(p, r)$ is continuous for $r = r_0$, (cf. (3.11) of [8]). An upper bound on the absolute difference between the bounds in (2.12) and (2.13) is asymptotically $(p \to 0)$ given by

$$(2.14) \qquad UB - LB \leq f(p, r)\{(g+1)r - gr\} = rf(p, r) \to 2.$$

Hence for the corresponding relative cost and its bounds we get $2/N$ which is small and approaches zero if $N \to \infty$. Thus these bounds (divided by N) virtually determine the function $H_D(N)/N$. If we assume that r/N is bounded as a function of p (say, by 1) then $(g+1)/N \sim (1 + \frac{r}{N})$ is bounded and it follows for small p that the above bounds in (2.12) in (2.13) have a minimum at the same point r_0 and hence $H_D(N)$ also has a minimum at r_0 for small p.

The approximation to r_0 which uses the nearest integer to the right side of (2.4) is quite good for (say) $p < .3$ and makes a table of exact results hardly necessary. In fact, a quick check of Dorfman's Table I (with our approximation) indicates some errors for $p < .3$ and a simple computation of $f^*(p, r)$ shows that the values for $p = .07$ and .12 should be changed from 5 to 4 and from 4 to 3, respectively. These same errors also appear in Table 2 of [8].

When $1/\sqrt{p}$ is exactly an integer so that the sum of the first two terms is midway between two integers, it is generally desirable to check both of these values by substituting them in $f(p, r)$ in (2.1) and not depend on the sign of the third term in (2.4). However an exact calculation does show that for $p = .01$ the value $r_0 = 10$ is better for R_D and $r_0 = 11$ is better for R_D^*. A similar change and improvement under R_D^* was found for $p = .04$ (where we change from $r = 6$ to $r = 5$) and for $p = .25$ (where we change from $r = 3$ to $r = 2$) but not for $p = .09$ (where we still use $r = 4$), cf. Table I of Dorfman. This is an additional justification for going to a three-term approximation in (2.4); these three terms are also used for the generalized Dorfman procedures in the next section.

3. Generalized Dorfman Procedures.

We now return to our original problem with two types of tests (cf. section 1 above). Two generalized Dorfman procedures $R_{12}^{(D)}$ and $R_{11}^{(D)}$ are defined for this problem. These are to be compared with the trivial procedure of testing each unit twice separately, which requires $2N$ tests for all p_1, p_2. We define an item (or unit) to be defective (T_i) if it is defective with respect to test T_i $(i = 1, 2)$; a defective unit is defective with respect to both tests.

Both of our procedures (in this and subsequent sections) use inference whenever possible. The first procedure $R_{12}^{(D)}$ uses two different group sizes r_1 and r_2 and can only be used when N is finite. Under $R_{12}^{(D)}$ we first carry out all required T_1-tests using the common group size r_1 and then, using only the units that are defective (T_1) as the new starting number of units, we carry out all required T_2-tests using the common group size r_2.

It can be argued that the minimizing values of r_1 and r_2 are both approximately given by (2.4) with $p = p_1$ and $p = p_2$, respectively. The reason for this is that the only connection between the two parts of $R_{12}^{(D)}$ is the number of defectives (T_1) and this is independent of the procedure of finding these defectives. In using (2.4) we are assuming that the expected number of defectives (T_1), i.e., Np_1, will not be too small so that we can use the approximations of section 2 above.

We are assuming here that $p_1 \leq p_2$ and that better results can be obtained by carrying out all the T_1-tests first. This will be seen for procedure $R_{12}^{(D)}$ after (3.6) below.

We now obtain lower and upper bounds for $H_{12}^{(D)}(N)$ somewhat similarly to (2.12) and (2.13) above. For the lower bound we use the result (2.12) that for the T_1 tests alone the lower bound is $Nf(p_1, r_1)$ where r_1 is given by (2.4) with $p = p_1$, if p_1 is small. Let x denote the number of defectives (T_1) arising from the T_1-tests. Then using the value $r = r_2$ in $f(p_2, r)$ for all x units we obtain the overall lower bound

$$(3.1) \qquad H_{12}^{(D)}(N) \geq Nf(p_1, r_1) + \sum_{x=1}^{N} \binom{N}{x} p_1^x q_1^{N-x} x f(p_2, r_2)$$

$$= N\{f(p_1, r_1) + p_1 f(p_2, r_2)\},$$

where r_i is given by the nearest integer to the right side of (2.4) with $p = p_i$ $(i = 1, 2)$.

For the upper bound we treat the x units that are defective (T_1) as a single group for the T_2-tests. Hence, using (2.13) above and (157) of [6]

$$(3.2) \qquad H_{12}^{(D)}(N) \leq (g_1+1)(r_1+1)f(p_1, r_1+1) + \sum_{x=1}^{N} \binom{N}{x} p_1^x q_1^{N-x} x f(p_2, x) .$$

The summation S in (3.2) can be computed exactly and we obtain with the help of (2.1)

$$(3.3) \qquad S = 1 - q_1^N\left(1 - \frac{p_2}{q_2}\right) - (p_1 q_2 + q_1)^N \frac{p_2}{q_1} + Np_1\left\{1 - q_2(p_1 q_2 + q_1)^{N-1}\right\}.$$

For $p_1 \to 0$ the value of $S \to 0$ and hence for small p_1 using (3.2) we can bound our result by

$$(3.4) \qquad H_{12}^{(D)}(N) \leq \left(\frac{N}{r_1} + 1\right)(r_1 + 1)f(p_1, r_1+1) \sim Nf(p_1, r_1) + 2 \sim 2g_1 + 4$$

(cf. (3.11) of [8]). The lower bound in (3.1) for $p_1 \to 0$ approaches

$2g_1 + 2$ and the difference between these bounds on $H_{12}^{(D)}(N)$ for p_1 small is 4. Thus $H_{12}^{(D)}(N)/N$ is virtually determined for p_1 small as before.

Using the notation $H_D(N|p_i)$ $(i = 1, 2)$ and $H_{12}^{(D)}(N|p_1, p_2)$ to exhibit the p-values, we obtain (as above) for the exact expected number of tests required by procedure $R_{12}^{(D)}$

$$H_{12}^{(D)}(N|p_1, p_2) = H_D(N|p_1) + \sum_{x=1}^{N} \binom{N}{x} p_1^x q_1^{N-x} H_D(x|p_2), \tag{3.5}$$

where the exact value of $H_D(N|p)$ is given in (2.8). It is this value (3.5) that we obtained bound for in (3.1) and (3.2) above.

In analogy with the method of finding r_0 for procedures R_D and R_D^*, we can also use the lower bound (3.1) to approximate r_1. For p_1 small, the equation determining r_1 is exactly the same as in (2.3) with $p = p_1$; this result does not depend on p_2.

As described in Section 2 the nearest integer r_1 to the right side of (2.4) for $p = p_1$ [or the solution of (2.3)] is the nominal value and the use of one of the two alternative partitions in Section 2 may alter this value r_1 if N is finite and not a multiple of r_1. Using the (random) number X of units that are defective (T_1) as a starting value and the nominal value r_2 obtained from (2.3) or (2.4) with $p = p_2$, we again use one of the two alternative partitions of Section 2. The first term of the result (2.5) with $p = p_1$ is also a good approximation for the procedure $R_{12}^{(D)}$ if only p_1 is small since it is $2N\sqrt{p_1}$ and the expected number of T_2-tests is of order $2Np_1\sqrt{p_2}$, which is smaller. However if p_1 and p_2 are both small then we have to add all three terms of the T_1-tests to the leading term of the T_2-tests, obtaining

$$(3.6) \qquad H_{12}^{(D)}(N) \sim 2N\sqrt{p_1}\left(1 - \frac{\sqrt{p_1}}{4} - \frac{5p_1}{8}\right) + 2Np_1\sqrt{p_2}$$

$$= 2N\sqrt{p_1}\left(1 - \frac{\sqrt{p_1}}{4} - \frac{5p_1}{8} + \sqrt{p_1 p_2}\right).$$

If $p_1 = p_2$ then the last two terms combine; otherwise the last term $\sqrt{p_1 p_2}$ has an order of magnitude between $\sqrt{p_1}$ and p_1, since $p_1 \leq p_2$. We also kept three terms in (2.4) and (2.5), in order to be able to write (3.6) in the above form and to be able to combine the last two terms if $p_1 = p_2$.

To see that it is better to carry out the T_1-tests first, we can use (3.6) and compare it with the result obtained by interchanging p_1 and p_2. Then $R_{12}^{(D)}$ as defined is better if

$$(3.7) \qquad 2(p_2 - p_1) + 5(p_2^{3/2} - p_1^{3/2}) < 8(1 + \sqrt{p_1 p_2})(\sqrt{p_2} - \sqrt{p_1})$$

or equivalently if

$$(3.8) \qquad 2(\sqrt{p_2} + \sqrt{p_1}) + 5p_2 - 3\sqrt{p_1 p_2} + 5p_1 < 8$$

which clearly holds even if only p_1 is sufficiently small.

The following lemma is used in procedure R_{11} below and also in procedure R_{10} developed in Section 4.

Lemma 1.

Let $s \leq m$ denote the size of a suspicious set $\dot{S}_m$, i.e., a subset of some defective set of size m. The conditional probability that a random sample of size $x \leq s$ contains no defective units given that it comes from S_m is

$$(3.9) \qquad Q(x, m) = \frac{q^x - q^m}{1 - q^m} \qquad (0 \leq x < m)$$

and does not depend on s.

Proof:

The probability that this sample is free of defectives is

$$(3.10)\qquad Q(x, m) = \sum_{y=1}^{m} \sum_{j} \frac{\binom{m}{y}p^{y}q^{m-y}}{1 - q^{m}} \frac{\binom{m-y}{s-j}\binom{y}{j}}{\binom{m}{s}} \frac{\binom{s-j}{x}}{\binom{s}{x}} ,$$

where j ranges from $\max(0, s-m+y)$ to $\min(s, y)$. Simplifying and summing the resulting hypergeometric gives

$$(3.11)\qquad Q(x, m) = \frac{q^{x}}{1 - q^{m}} \sum_{j=1}^{m} p^{y}q^{m-x-y}\binom{m-x}{y},$$

since we get a zero summand for any $j > s - x$. Since the sum in (3.11) is zero for $y > m - x$ we sum the resulting binomial to $y = m - x$ and obtain the result in (3.9).

The second generalized Dorfman procedure $R_{11}^{(D)}$ defines a single group-size $r = r_{11}$ and each group of size r that does not pass its T_1-test undergoes a T_2-test (on this entire group) as the very next test. If that also fails then each of the r units separately undergoes both a T_1-test and a T_2-test. Inference saves exactly one test if and only if $r_{11} \geq 2$ and either i) the first $r - 1$ of a group that is defective (T_1) are all good (T_1), or ii) every one of the r units is defective (T_1), the entire set of size r is defective (T_2), and the first $r - 1$ units are all good (T_2). These two events i) and ii) are mutually disjoint. The optimal value of r will be determined after deriving expressions for $H_{11}^{(D)}(N)$ and $H_{11}^{(D)}(r)$. If a set of size r passes its first set or its second test then we are done with it; otherwise each unit that is defective (T_1) requires two additional tests and every unit that is good (T_1) requires only one. Hence we obtain for $r \geq 2$

$$(3.12)\qquad H_{11}^{(D)}(r) = q_1^r + 2(1 - q_1^r) + (1 - q_2^r)\sum_{i=1}^{r}\binom{r}{i}p_1^i q_1^{r-i}\{2i + (r-i)\}$$

$$- p_1 q_1^{r-1} - p_1^r p_2 q_2^{r-1}$$

$$= 2 - q_1^r + r(1 - q_2^r)(p_1 + 1 - q_1^r) - p_1 q_1^{r-1} - p_1 p_2 (p_1 q_2)^{r-1},$$

and for $r = 1$ the result is $H_{11}^{(D)}(1) = 1 + p_1$. Hence the expected number of tests per unit classified, which we want to minimize, is for $N = kr$ and $r \geq 2$

$$(3.13)\qquad \frac{H_{11}^{(D)}(r)}{r} \doteq f(p_1, p_2, r) = (1 - q_2^r)(p_1 + 1 - q_1^r) + \frac{2 - q_1^r - p_1 q_1^{r-1}}{r},$$

where we have dropped the last (inference) term in (3.12) which has a factor p_1^r. Differentiation with respect to r, setting the result equal to zero and looking only for the leading terms as $p_1 \to 0$, we obtain

$$(3.14)\qquad r^2 p_1 + r^3 p_1 p_2 \sim 1$$

as the approximate equation defining r. For $p_2 = cp_1^{\varepsilon}$ a further analysis yields three cases according as $\frac{1}{2} < \varepsilon \leq 1$, $\varepsilon = \frac{1}{2}$, $0 \leq \varepsilon < \frac{1}{2}$. We are interested only in Case 1 ($\frac{1}{2} < \varepsilon \leq 1$) which is very roughly characterized by saying that p_1 and p_2 are close to each other as $p_1 \to 0$. Then the first term on the left of (3.14) dominates the second and we again find that $r \sim 1/\sqrt{p_1}$. The same result is obtained for the two subcases given by $\frac{1}{2} < \varepsilon < 1$ and $\varepsilon = 1$, namely that (3.14) gives two correct terms and these are

$$(3.15)\qquad r \sim \frac{1}{\sqrt{p_1}} - \frac{c}{2p_1^{1-\varepsilon}} = \frac{1}{\sqrt{p_1}} - \frac{p_2}{2p_1} \; .$$

If we substitute this back in (3.12), we obtain

$$(3.16)\qquad H_{11}^{(D)}(N) \sim N(\sqrt{p_1} + \tfrac{3}{2}p_2 + p_1).$$

If we compare the result (3.16) for $R_{11}^{(D)}$ with the result $2N\sqrt{p_1}$ for $R_{12}^{(D)}$ in (3.6) then we note that for $\epsilon < 1$, the procedure $R_{11}^{(D)}$ is better when

$$(3.17)\qquad \tfrac{3}{2}p_2 < \sqrt{p_1} \quad \text{or} \quad p_1^{2\epsilon-1} < \frac{4}{9c^2}\;.$$

Since $2\epsilon - 1 > 0$ in this case of interest, $R_{11}^{(D)}$ will be better than $R_{12}^{(D)}$ for p_1 sufficiently small. For $\epsilon = 1$, we add in the term p_1 in (3.16) but the same result holds. It is also clear that $R_{12}^{(D)}$ will be better if p_1 and p_2 are far enough apart (for example if $p_1 \to 0$ and p_2 remains fixed). This corroborates the roughly stated conclusion that $R_{11}^{(D)}$ is better when p_1 and p_2 are close together and $R_{12}^{(D)}$ is better when they are far apart.

4. More Efficient Procedures.

In this section we develop some procedures that are similar to the procedure R_1 [6], that is applicable when only one test is applied to the units. We again assume $p_1 \leq p_2$ and therefore start with the T_1-tests. The procedure R_{12} corresponds to the case in which we make all the T_1-tests first and, using D to denote the number of defectives (T_1) that arise, we then start with D defectives and make all the T_2-tests. Let d denote the current number of units that were shown to be defective (T_1), so that $0 \leq d \leq D$. The entire procedure R_{12} will now be defined as a single algorithm using primes for the T_1-tests and no primes for the T_2-tests. Our notation is similar to that in [6]; we let $H'(n|d)$ denote the expected total number of additional tests required if we currently have n binomial units not yet classified (T_1) and d units were shown to be defective (T_1). $G'(m, n|d)$ denotes the expected total number of additional tests required if we currently have n units not yet classified (T_1), m of these belong to a defective subset (i.e., a subset that is known to contain at least one defective unit) and d is as above.

For $n \geq 1$ and $d \geq 0$

$$(4.1) \qquad H'(n|d) = 1 + \min_{1 \leq x \leq n} \{q_1^x H'(n-x|d) + (1-q_1^x) G'(x, n|d)\}.$$

For $2 \leq m \leq n$ and $d \geq 0$

$$(4.2) \qquad G'(m, n|d) = 1 + \min_{1 \leq x < m} \{(\frac{q_1^x - q_1^m}{1 - q_1^m}) G'(m-x, n-x|d) + (\frac{1 - q_1^x}{1 - q_1^m}) G'(x, n|d)\}.$$

For $n \geq 1$ and $d \geq 0$

$$(4.3) \qquad G'(1, n|d) = H'(n-1|d+1) \; ; \; H'(0|d) = H(d).$$

For $n \geq 1$

$$(4.4) \qquad H(n) = 1 + \min_{1 \leq x \leq n} \{q_2^x H(n-x) + (1-q_2^x) G(x, n)\}.$$

For $2 \leq m \leq n$

$$(4.5) \qquad G(m, n) = 1 + \min_{1 \leq x < m} \{(\frac{q_2^x - q_2^m}{1 - q_2^m}) G(m-x, n-x) + (\frac{1 - q_2^x}{1 - q_2^m}) G(x, m)\}.$$

For $n \geq 1$

$$(4.6) \qquad G(1, n) = H(n-1) \; ; \; H(0) = 0.$$

To distinguish these G and H functions from others for different procedures we will write them as G_{12} and H_{12} when making comparisons. As in the case of the Dorfman-type procedures we expect this procedure to be highly efficient when p_1 and p_2 are far apart in the sense described in Section 3.

One good reason for stressing this procedure is that it is a combination of two independent R_1-type procedures. It follows that the first part of R_{12} is exactly R_1 and even the second part enjoys all the same properties as R_1 if these properties do not depend on the initial population size N. In fact we can write the answer $H_{12}(N|p_1, p_2)$ in terms of $H_1(N|p_i)$ $(i = 1, 2)$ for procedure R_1 given in [6]. As in Section 3 we obtain

$$H_{12}(N|p_1, p_2) = H_1(N|p_1) + \sum_{j=1}^{N} \binom{N}{j} p_1^j q_1^{N-j} H_1(j|p_2). \tag{4.7}$$

For $N = 2$ we use the results in [6] and obtain

$$H_{12}(N|p_1, p_2) = \begin{cases} 4 - 2q_1 & \text{for } q_2 \leq q_1 < q_0 \\ 1 + p_1(4+q_1) & \text{for } q_2 \leq q_0 \leq q_1 \\ 1 + p_1(5-p_1q_2 - p_1q_2^2) & \text{for } q_0 < q_2 \leq q_1 \end{cases}, \tag{4.8}$$

where $q_0 = (\sqrt{5} - 1)/2 = .618\ldots$ and equality holds at any points that two or all three regions (at the far right of (4.8)) may have in common; hence we can randomize between the corresponding procedures when equalities hold.

Thus, in essence, we have available many results about procedure R_{12}. The corresponding numerical results for most of the procedures in this paper are not yet available.

The procedure R_{11} starts with a T_1-test on some subset of size x and, if it fails, this is followed by a T_2-test on the very same subset. Let the symbol $G(m, n)$ denote the expected number of additional tests required when n units are still unclassified and a subset of size m contains at least one defective (T_1) and at least one defective (T_2). Then $G(0, m, n)$ indicates that a subset of size m contains at least one unit that is defective (T_2), etc. Let $G(r, {}^{m}_{0}, n)$ with $r \leq \min(m, n)$ denote the same expectation if (currently) among the n unclassified units there is a defective (T_1) subset of size r (i.e., it contains at least one unit that is defective (T_1)) and these r units were previously among a set of m units that was known to contain at least one that was defective (T_2); then $G(m, {}^{m}_{0}, n) = G(m, n)$. The symbol $G(0, {}^{m}_{s}, n)$ with $s \leq m$ denotes the same expectation if (currently) a subset of size s (called suspicious) was previously included among a defective (T_2) subset of size m, i.e., a set of size m that contained at least one unit that is defective (T_2); then $G(0, {}^{m}_{m}, n) = G(0, m, n)$ and $G(0, {}^{m}_{0}, n) = G(0, 0, n) = G({}^{m}_{0}, 0, n)$ and $G({}^{m}_{m}, 0, n) = G(m, 0, n)$.

We can now write the algorithm for procedure R_{11} in 5 equations plus boundary conditions. For $n \geq 1$

$$(4.9) \qquad H(n) = \underset{1 \leq x \leq n}{\text{Min}} \left(q_1^x \{1 + H(n-x)\} + (1-q_1^x) q_2^x \{2 + H(n-x)\} + (1-q_1^x)(1-q_2^x)\{2 + G(x, n)\} \right).$$

For $2 \leq m \leq n$

$$(4.10) \qquad G(m, n) = \underset{1 \leq x < m}{\text{Min}} \left(\left(\frac{q_1^x - q_1^m}{1 - q_1^m}\right)\{1 + G(m-x, {}^{m}_{0}, n-x)\} + \left(\frac{1-q_1^x}{1-q_1^m}\right)\left(\frac{q_2^x - q_2^m}{1 - q_2^m}\right)\{2 + G(0, m-x, n-x)\} + \left(\frac{1-q_1^x}{1-q_1^m}\right)\left(\frac{1-q_2^x}{1-q_2^m}\right)\{2 + G(x, n)\} \right).$$

For $2 \leq m \leq n$

$$(4.11)\qquad G(0,\ m,\ n) = \underset{1\le x<m}{\text{Min}} \Bigg(q_1^x\{1 + G(0,\ \genfrac{}{}{0pt}{}{m}{m-x},\ n-x)\}$$

$$+ (1-q_1^x)\left(\frac{q_2^x - q_2^m}{1 - q_2^m}\right)\{2 + G(0,\ m-x,\ n-x)\}$$

$$+ (1-q_1^x)\left(\frac{1-q_2^x}{1-q_2^m}\right)\{2 + G(x,\ n)\}\Bigg).$$

For $1 \le s < m$ and $s \le n$

$$(4.12)\qquad G(0,\ \genfrac{}{}{0pt}{}{m}{s},\ n) = \underset{1\le x\le s}{\text{Min}} \Bigg(q_1^x\{1 + G(0,\ \genfrac{}{}{0pt}{}{m}{s-x},\ n-x)\}$$

$$+ (1-q_1^x)\left(\frac{q_2^x - q_2^m}{1 - q_2^m}\right)\{2 + G(0,\ \genfrac{}{}{0pt}{}{m-x}{s-x},\ n-x)\}$$

$$+ (1-q_1^x)\left(\frac{1-q_2^x}{1-q_2^m}\right)\{2 + G(x,\ n)\}\Bigg).$$

For $2 \le r \le \min(m,\ n)$

$$(4.13)\qquad G(r,\ \genfrac{}{}{0pt}{}{m}{0},\ n) = \underset{1\le x<r}{\text{Min}} \Bigg(\left(\frac{q_1^x - q_1^r}{1 - q_1^r}\right)\{1 + G(r-x,\ \genfrac{}{}{0pt}{}{m}{0},\ n-x)\}$$

$$+ \left(\frac{1-q_1^x}{1-q_1^r}\right)\left(\frac{q_2^x - q_2^m}{1 - q_2^m}\right)\{2 + G(0,\ \genfrac{}{}{0pt}{}{m-x}{r-x},\ n-x)\}$$

$$+ \left(\frac{1-q_1^x}{1-q_1^r}\right)\left(\frac{1-q_2^x}{1-q_2^m}\right)\{2 + G(x,\ n)\}\Bigg).$$

The boundary conditions are

$$G(1,\ n) = H(n-1)\ ;\quad H(0) = 0,$$

$$(4.14)\qquad G(0,\ 1,\ n) = 1 + H(n-1) = G(1,\ \genfrac{}{}{0pt}{}{m}{0},\ n)\quad \text{for}\quad m > 1,$$

$$G(0,\ \genfrac{}{}{0pt}{}{m}{0},\ n) = H(n) = G(1,\ \genfrac{}{}{0pt}{}{1}{0},\ n).$$

For $N = 2$ the above algorithm for procedure R_{11} yields

$$(4.15)\qquad H(2) = \text{Min}\begin{cases} 2 + 2p_1 & \text{if } x = 1 \\ 1 + p_1(5+2q_1 - q_2 + q_1q_2 - 3q_2^2 - 2q_1q_2^2) & \text{if } x = 2 . \end{cases}$$

Here we take $x = 1$ or $x = 2$ at the outset according to which of the above two expressions is smaller. The path of equality is the boundary curve, which when solved for q_2 as a quadratic yields

$$(4.16)\qquad q_2 = \frac{\sqrt{p_1^4 + 4p_1(3+2q_1)(2-q_1 - 2q_1^2)} - p_1^2}{2p_1(3+2q_1)} ,$$

which goes through the points (.594, .594)(.781, 0), (.618, .575) and (0.667).

If we compare procedures R_{11} and R_{12} for $N = 2$ then we find that R_{11} is preferred in the corner where q_1 and q_2 are both near 1, R_{12} is preferred when q_1 is near one and q_2 is near zero and equality holds when both q_1 and q_2 are near zero. The region for preference to use R_{12} is connected but the region of preference for R_{11} is not.

As a final procedure we develop an even more efficient procedure (denoted by R_{10}) in the class of R_1-type procedures. In this case we sacrifice simplicity to get something more efficient. We believe that this procedure is optimal in the class of procedures of R_1-type, but as shown in [6], [4] and [5] for the case of 1 test this will generally not be optimal in the class of all procedures. On the other hand it is clear that to gain more efficiency we have to introduce more complication and it may become necessary to quantify this notion of complication; we prefer to avoid this in the present paper.

In the procedure R_{10} we collect units that are defective $(T_i)(i = 1, 2)$ separately and the notation $G(m, m', n|d_1, d_2) = G(m, m', n)$ indicates that we currently have d_i units that are defective (T_i) $(i = 1, 2)$; in the algorithm this information will be kept but may not be shown below except in some boundary conditions.

In most equations the value of d_1 will change to $d_1 + 1$ if $x = 1$ is the option selected and the unit is shown to be defective (T_1); similarly $d_2 \to d_2 + 1$ if $y = 1$ is the option selected and the unit is shown to be defective (T_2). Otherwise the notation is as before, m and m' denoting the sizes of two disjoint subsets that contain at least one unit that is defective (T_1) and (T_2), respectively. For $m = m' = 0$ we say we are in an H-situation and write in overlapping terminology

$$G(0, 0, n|d_1, d_2) = H(0, 0, n|d_1, d_2) \quad (n \geq d_1 + d_2) \tag{4.17}$$
$$= \begin{cases} H_1(d_1|p_2) & \text{if } d_2 = 0 \text{ and } n = d_1 \\ H_1(d_2|p_1) & \text{if } d_1 = 0 \text{ and } n = d_2 , \end{cases}$$

where the subscript 1 refers to procedure R_1 studied in [6] and n is the total (current) number of unclassified units. The symbol $G(\begin{smallmatrix} r \\ 0 \end{smallmatrix}, \begin{smallmatrix} m \\ s \end{smallmatrix}, n|d_1, d_2) = G(\begin{smallmatrix} r \\ 0 \end{smallmatrix}, \begin{smallmatrix} m \\ s \end{smallmatrix}, n)$ with $s < \min(r, m)$ indicates that s units came from a subset of size m that contained at least one unit that is defective (T_2) and also that the same s units came from a subset of size r that contained at least one unit that is defective (T_1); this information determines the probability weights in the recursion formula. Consistent with the notation for procedure R_{11}, we use $G(m, \begin{smallmatrix} r \\ s \end{smallmatrix}, n)$ with $m + s \leq r$ to indicate two disjoint sets of size m and s; the first one is defective (T_1) and both sets were previously included in a common set of size r that was defective (T_2).

For $n \geq d_1 + d_2 + 1$

$$(4.18) \qquad H(0, 0, n) = 1 + \min_{1 \leq x \leq n-d_1-d_2} \{q_1^x H(0, 0, n-x) + (1-q_1^x) G(x, 0, n)\};$$

when $x = 1$ the value of d_1 on the left side of (4.18) changes to $d_1 + 1$ in $G(x, 0, n)$. For $2 \leq \min(m, m')$

$$(4.19) \qquad G(m, m', n) = 1 + \min \begin{pmatrix} \min_{1 \leq x < m} \{Q_1(x,m)G(m-x,m',n-x) + P_1(x,m)G(x,m',n)\}, \\ \min_{1 \leq y < m'} \{Q_2(y,m')G(m,m'-y,n-y) + P_2(y,m')G(m,y,n)\} \end{pmatrix},$$

where $Q_1(x, m) = 1 - P_1(x, m)$ is given by $(q_1^x - q_1^m)/(1-q_1^m)$ and similarly for $Q_2(y, m') = 1 - P_2(y, m')$. For $2 \leq m \leq n$

$$(4.20) \qquad G(m, 0, n) = 1 + \min \begin{pmatrix} \min_{1 \leq x < m} \{Q_1(x,m)G(m-x,0,n-x) + P_1(x,m)G(x,0,n)\}, \\ \min_{1 \leq y < m} \{q_2^y G(\binom{m}{m-y},0,n-y) + (1-q_2^y)G(\binom{m}{m-y},y,n)\} \end{pmatrix},$$

where we allow either a T_1-test on any <u>proper</u> subset of the defective set of size m or a T_2-test on <u>any</u> subset of this defective set. For $1 \leq s < r$

$$(4.21) \qquad G(\binom{r}{s}, 0, n) = 1 + \min \begin{pmatrix} \min_{1 \leq x \leq s} \{Q_1(x,s)G(\binom{r-x}{s-x},0,n-x) + P_1(x,s)G(x,0,n)\}, \\ \min_{1 \leq y \leq s} \{q_2^y G(\binom{r}{s-y},0,n-y) + (1-q_2^y)G(\binom{r}{s-y},y,n)\} \end{pmatrix}.$$

For $1 \leq s < r$ and $2 \leq m \leq r-s$

$$(4.22) \qquad G(\binom{r}{s}, m, n) = 1 + \min \begin{pmatrix} \min_{1 \leq x \leq s} \{Q_1(x,r)G(\binom{r-x}{s-x},m,n-x) + P_1(x,r)G(x,m,n)\}, \\ \min_{1 \leq y < m} \{Q_2(y,m)G(\binom{r}{s},m-y,n-y) + P_2(y,m)G(\binom{r}{s},y,n)\} \end{pmatrix}.$$

For $2 \leq m < r$

$$(4.23) \qquad G(\binom{r}{0}, m, n) = 1 + \min \begin{pmatrix} \min_{1 \leq x \leq m} \{Q_1(x,r)G(\binom{r-y}{0},\binom{m}{m-x},n-x) + P_1(x,n)G(x,\binom{m}{m-x},n)\}, \\ \min_{1 \leq y < m} \{Q_2(y,m)G(\binom{r}{0},m-y,n-y) + P_2(y,m)G(\binom{r}{m-y},y,n)\} \end{pmatrix}.$$

For $m = r$ in (4.23) we write the same as above with $1 \leq x < m$ and $G(m\text{-}x, \genfrac{}{}{0pt}{}{m}{0}, n\text{-}x)$ for the upper left G-function on the right side of (4.23). For $2 \leq m \leq n$

$$(4.24) \qquad G(0, m, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \underset{1\le x\le m}{\mathrm{Min}} \{q_1^x G(0, \genfrac{}{}{0pt}{}{m}{m-x}, n-x) + (1-q_1^x) G(x, \genfrac{}{}{0pt}{}{m}{m-x}, n)\}, \\ \\ \underset{1\le y<m}{\mathrm{Min}} \{Q_2(y,m) G(0, m-y, n-y) + P_2(y,m) G(0, y, n)\} \end{array} \right).$$

For $1 \leq s < r$

$$(4.25) \qquad G(\genfrac{}{}{0pt}{}{r}{s}, 1, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \{Q_1(1,r) G(\genfrac{}{}{0pt}{}{r-1}{s}, 0, n-1) + P_1(1,r) G(0,0,n-1)\}, \\ \underset{1\le x\le s}{\mathrm{Min}} \{Q_1(x,r) G(\genfrac{}{}{0pt}{}{r-x}{s-x}, 1, n-x) + P_1(x,r) G(x,1,n)\} \end{array} \right).$$

For $1 \leq s < r$

$$(4.26) \qquad G(1, \genfrac{}{}{0pt}{}{r}{s}, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \{Q_2(1,r) G(0, \genfrac{}{}{0pt}{}{r-1}{s}, n-1) + P_2(1,r) G(0,0,n-1)\}, \\ \underset{1\le y\le s}{\mathrm{Min}} \{Q_2(y,r) G(1, \genfrac{}{}{0pt}{}{r-y}{s-y}, n-y) + P_2(y,r) G(1,y,n)\} \end{array} \right).$$

For $1 \leq s < r$

$$(4.27) \qquad G(0, \genfrac{}{}{0pt}{}{r}{s}, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \underset{1\le x\le s}{\mathrm{Min}} \{q_1^x G(0, \genfrac{}{}{0pt}{}{r}{s-x}, n-x) + (1-q_1^x) G(x, \genfrac{}{}{0pt}{}{r}{s-x}, n)\}, \\ \\ \underset{1\le y\le s}{\mathrm{Min}} \{Q_2(y,r) G(0, \genfrac{}{}{0pt}{}{r-y}{s-y}, n-y) + P_2(y,r) G(0,y,n)\} \end{array} \right).$$

For $1 \leq s < r$ and $2 \leq m \leq r\text{-}s$

$$(4.28) \qquad G(m, \genfrac{}{}{0pt}{}{r}{s}, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \underset{1\le x<m}{\mathrm{Min}} \{Q_1(x,m) G(m-x, \genfrac{}{}{0pt}{}{r}{s}, n-x) + P_1(x,m) G(x, \genfrac{}{}{0pt}{}{r}{s}, n)\}, \\ \\ \underset{1\le y\le s}{\mathrm{Min}} \{Q_2(y,r) G(m, \genfrac{}{}{0pt}{}{r-y}{s-y}, n-y) + P_2(y,r) G(m,y,n)\} \end{array} \right).$$

For $2 \leq m < r$

$$(4.29) \qquad G(m, \genfrac{}{}{0pt}{}{r}{0}, n) = 1 + \mathrm{Min}\left(\begin{array}{l} \underset{1\le x<m}{\mathrm{Min}} \{Q_1(x,m) G(m-x, \genfrac{}{}{0pt}{}{r}{0}, n-x) + P_1(x,m) G(x, \genfrac{}{}{0pt}{}{r}{m-x}, n)\}, \\ \\ \underset{1\le y\le m}{\mathrm{Min}} \{Q_2(y,r) G(\genfrac{}{}{0pt}{}{m}{m-y}, \genfrac{}{}{0pt}{}{r-y}{0}, n-y) + P_2(y,r) G(\genfrac{}{}{0pt}{}{m}{m-y}, y, n)\} \end{array} \right).$$

For $m = r$ in (4.29) we write the same as above with $1 \leq y < m$ and $G(\genfrac{}{}{0pt}{}{m}{0}, m\text{-}y, n\text{-}y)$ for the lower left G-function on the right side of (4.29).

For $1 \leq s < \min(r, m)$

$$(4.30)\qquad G\left(\genfrac{}{}{0pt}{}{r}{0}, \genfrac{}{}{0pt}{}{m}{s}, n\right) = 1 + \text{Min}\begin{pmatrix} \underset{1\le x\le s}{\text{Min}} \{Q_1(x,r)G(\genfrac{}{}{0pt}{}{r-x}{0}, \genfrac{}{}{0pt}{}{m}{s-x}, n-x) + P_1(x,r)G(x, \genfrac{}{}{0pt}{}{m}{s-x}, n)\}, \\ \underset{1\le y\le s}{\text{Min}} \{Q_2(y,m)G(\genfrac{}{}{0pt}{}{r}{0}, \genfrac{}{}{0pt}{}{m-y}{s-y}, n-y) + P_2(y,m)G(\genfrac{}{}{0pt}{}{r}{0}, y, n)\} \end{pmatrix}.$$

For $1 \le s < \min(r, m)$

$$(4.31)\qquad G\left(\genfrac{}{}{0pt}{}{m}{s}, \genfrac{}{}{0pt}{}{r}{0}, n\right) = 1 + \text{Min}\begin{pmatrix} \underset{1\le x\le s}{\text{Min}} \{Q_1(x,m)G(\genfrac{}{}{0pt}{}{m-x}{s-x}, \genfrac{}{}{0pt}{}{r}{0}, n-x) + P_1(x,m)G(x, \genfrac{}{}{0pt}{}{r}{0}, n)\}, \\ \underset{1\le y\le s}{\text{Min}} \{Q_2(y,r)G(\genfrac{}{}{0pt}{}{m}{s-y}, \genfrac{}{}{0pt}{}{r-y}{0}, n-y) + P_1(y,r)G(\genfrac{}{}{0pt}{}{m}{s-y}, y, n)\} \end{pmatrix}.$$

When we reach a situation in which every unit is either classified as defective (T_1) (by itself) or defective (T_2) (by itself) then we apply the R_1 procedure to each of the two piles (of size d_1 and d_2, respectively) and hence we write

$$(4.32)\qquad H(0, 0, d_1 + d_2) = H_1(d_1|p_2) + H_1(d_2|p_1).$$

Then the continuation as for procedure R_1 in [6] is given for $i = 1, 2$ by: for $n \ge 1$

$$(4.33)\qquad H_1(n|p_i) = 1 + \underset{1\le x\le n}{\text{Min}} \{q_i^x H_1(n-x|p_i) + (1-q_i^x)G_1(x,n|p_i)\},$$

for $2 \le m \le n$

$$(4.34)\qquad G_1(m, n|p_i) = 1 + \underset{1\le x< m}{\text{Min}} \{Q_i(x,m)G_1(m-x,n-x|p_i) + P_i(x,m)G_1(x,m|p_i)\}$$

with boundary conditions for $i = 1, 2$

$$(4.35)\qquad H_1(0|p_i) = 0 ; G(1, n|p_i) = H(n-1|p_i).$$

The remaining boundary conditions for procedure R_{01} are given by (4.17) and (4.32) and the following:

$$G(1,\ m,\ n|d_1,\ d_2) = G(0,\ m,\ n|d_1+1,\ d_2)\ ,$$
$$G(m,\ 1,\ n|d_1,\ d_2) = G(m,\ 0,\ n|d_1,\ d_2+1)\ ,$$
$$G(^m_m,\ 0,\ n) = G(m,\ 0,\ n)\ ;\ G(0,\ ^m_m,\ n) = G(0,\ m,\ n)\ ,$$
$$G(^m_0,\ 0,\ n) = G(0,\ ^m_0,\ n) = G(^r_0,\ ^m_0,\ n) = G(0,\ 0,\ n)\ ,$$
$$G(^1_0,\ 1,\ n) = G(1,\ ^1_0,\ n) = G(0,\ 0,\ n-1)\ , \tag{4.36}$$
$$G(^r_0,\ 1,\ n) = G(1,\ ^r_0,\ n) = 1 + G(0,\ 0,\ n-1) \quad \text{for} \quad r \geq 2\ ,$$
$$G(^r_0,\ ^m_r,\ n) = G(r,\ ^m_0,\ n)\ ;\ G(^m_r,\ ^r_0,\ n) = G(^m_0,\ r,\ n)\ ,$$
$$G(^r_0,\ ^m_m,\ n) = G(^r_0,\ m,\ n)\ ;\ G(^m_m,\ ^r_0,\ n) = G(m,\ ^r_0,\ n)\ ,$$
$$H(0,\ 0,\ 0) = 0\ .$$

A numerical evaluation for procedure R_{10} has not yet been carried out for large n values. For the particular value $q_1 = q_2 = (\sqrt{5} - 1)/2 = .618\ldots$ with $N = 2$ the procedures R_{12} and R_{11} and the one-at-a-time procedure all give the same result $H(2) = 4 - 2q_0 = 4 - 2(.618) = 2.764$. The procedure R_{10} gives 2.674 for this case, a reduction of .09. It is not known how much reduction can be had for higher values of n, q_1 and q_2.

5. A Lower Bound on $H(N|R)$ for any Procedure R.

Under any of the above procedures (except possibly for R_{10}, which we discuss separately below), we find out for each unit which of the three following mutually exclusive and exhaustive classes it belongs to: good on test T_1, bad on test T_1 and good on test T_2, bad on both tests. Hence it follows that we could regard our problem as a classification problem with these three categories. The total uncertainty associated with N independent units with probabilities q_1, p_1q_2 and p_1p_2 for these categories is well known to be

(5.1) $$-N\{q_1 \log_2 q_1 + p_1 q_2 \log_2 p_1 q_2 + p_1 p_2 \log_2 p_1 p_2\} .$$

Since the maximum information or reduction in uncertainty for a single test with two possible outcomes is $\frac{1}{2}\log_2(\frac{1}{2}) + \frac{1}{2}\log_2(\frac{1}{2}) = 1$ it follows that (5.1) is a lower bound for $H(N|R)$ for any procedure R that classifies each unit into one of these three categories.

The procedure R_{01} could conceivably find out that a unit is good on T_2 before finding out its reaction to test T_1; then for such units we are using the partition: good on T_2, bad on T_2 but good on T_1, bad on both tests. To show that (5.1) is still a lower bound we have to show that this new partition has a larger uncertainty. Since on term $p_1 p_2$ is common, we have only to show that

(5.2) $$-\{q_2 \log_2 q_2 + p_2 q_1 \log_2 p_2 q_1\} > -\{q_1 \log_2 q_1 + p_1 q_2 \log_2 p_1 q_2\} .$$

Since $p_1 \leq p_2$ it follows that

(5.3) $$p_1 q_2 \leq \min(p_2 q_1, q_2) \leq \max(p_2 q_1, q_2) \leq q_1$$

and hence, since $p_1 q_2 + q_1 = p_2 q_1 + q_2 = 1 - p_1 p_2$, it follows from the monotonicity property of $\Sigma p_i \log_2 p_i$ that (5.2) holds. Then the lower bound (5.1) holds for any procedure R in our problem.

For $p_1 = .01$, $N = 100$ and a few different values of p_2 the resulting LB values are given in the following table.

TABLE 1

p_2	.01	.05	.10	.50	1.0
LB (cf. (5.1))	8.1602	8.3658	8.5484	9.0793	8.0794
$H(100\|R_{12})$	8.96	9.01	9.07	9.32	9.32
$LB(100\|R_{12}^{(D)})$(cf. (3.1))	19.6651	19.8885	20.0461	20.4704	20.4704
$H(100\|R_{12}^{(D)})$	20.12	20.17	20.23	20.47	20.47

It is easy to show that the maximum of LB as a function of p_2 occurs at $p_2 = .5$. For $N = 100$ the expected number of tests attained by procedures R_{12} and $R_{12}^{(D)}$ and also the lower bound $LB(100|R_{12}^{(D)})$ on the right side of (3.1) are shown in the table above. The expectations were obtained by using a Poisson approximation with $Np_1 = \lambda = 1$ to the binomial probabilities in (3.5). The lower bound (LB) values are useful and interesting for $p_2 \leq .5$. We note that in this range R_{11} improves on R_{12} for $p_1 = p_2$, where the difference $H(N|R_{12})$-LB is somewhat larger. The extent of this improvement has not been evaluated. Table 1 shows that for small p_1 the results are essentially independent of p_2, when $p_1 \leq p_2$.

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REFERENCES

[1] Blumenthal, S., Kumar, S. and Sobel, M. (1971). Symmetric Binomial Group-Testing with 3 Outcomes. Purdue Symposium on Decision Procedures, Academic Press, New York.

[2] Dorfman, R. (1943). The detection of defective members of large populations. Ann. Math. Statist. **14** 436-440.

[3] Feller, W. (1968). An Introduction to Probability Theory and its Applications, 3rd ed. John Wiley, New York.

[4] Sobel, M. (1960). Group-testing to classify all defectives in a binomial sample. A contribution in Information and Decision Processes. McGraw-Hill Book Co., New York.

[5] _________ (1967). Optimal group-testing. Proceedings of the Colloquium on Information Theory (Bolyai Math. Soc., Debrecen, Hungary) 411-488.

[6] Sobel, M. and Groll, P. A. (1959). Group-testing to eliminate efficiently all defectives in a binomial sample. Bell System Tech. Journal **38** 1179-1252.

[7] _________ (1966). Binomial group testing with an unknown proportion of defectives. Technometrics **8** 631-656.

[8] Watson, G. S. (1961). A study of group-screening method. Technometrics **3** 371-388.